Convex compactness and its applications

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Abstract. The concept of convex compactness, weaker than the classical notion of compactness, is introduced and discussed. It is shown that a large class of convex subsets of topological vector spaces shares this property and that is can be used in lieu of compactness in a variety of cases. Specifically, we establish convex compactness for certain familiar classes of subsets of the set of positive random variables under the topology induced by convergence in probability.

Two applications in infinite-dimensional optimization - attainment of infima and a version of the Minimax theorem - are given. Moreover, a new fixed-point theorem of the Knaster-Kuratowski-Mazurkiewicz-type is derived and used to prove a general version of the Walrasian excess-demand theorem.

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1 Introduction

The raison d'être of the present paper is to provide the community with a new formulation and a convenient toolbox for a number of interrelated ideas centered around compactness substitutes in infinite-dimensional spaces that occur frequently in mathematical economics and finance. Our goal is to keep the presentation short and to-the-point and derive a number of (still quite general) mathematical theorems which we believe will find their use in the analysis of various applied problems in finance and economics. For this reason, we do not include worked-out examples in specific economic models, but paint a broader picture and point to some common areas of interest (such as infinite-dimensional optimization or general-equilibrium theory) where we believe our results can be successfully applied.

Compactness in infinite-dimensional topological vector spaces comes at a cost: either the size of the set, or the strength of the topology used have to be severely restricted. Fortunately, the full compactness requirement is often not necessary for applications, especially in mathematical economics where a substantial portion of the studied objects exhibits at least some degree of convexity. This fact has been heavily exploited in the classical literature, and refined recently in the work of Delbaen and Schachermayer (1994, 1999); Kramkov and Schachermayer (1999) and others in the field of mathematical finance (e.g., in the context of the fundamental theorem of asset pricing and utility maximization theory in incomplete markets). More specifically, the above authors, as well as many others, used various incarnations of the beautiful theorem of Komlós (1967) (see, also, Schwartz (1986)) to extract a convergent sequence of convex combinations from an arbitrary (mildly bounded) sequence of non-negative random variables (see, e.g., Nikišin (1971); Buhvalov and Lozanovskiĭ (1973) for some of the earliest references). In the context of convex optimization problems this procedure is often as versatile as the extraction of a true subsequence, but can be applied in a much larger number of situations.

The initial goal of this note is to abstract the precise property - called *convex compactness* - of the space $\mathbb{L}^0_+$ (with $\mathbb{L}^0_+$ denoting the set of all a.s.-equivalence classes of non-negative random variables, equipped with the complete metric topology of convergence in probability) which allows for the rich theory mentioned above. Our first result states that, in the class of convex subsets of $\mathbb{L}^0_+$, the convexly compact sets are precisely the bounded and closed ones. An independently-useful abstract characterization in terms or generalized sequences is given. We stress that the language of probability theory is used for convenience only, and that all of our results extend readily to abstract measure spaces with finite measures.

Recent research in the field of mathematical finance has singled out $\mathbb{L}^0_+$ as especially important - indispensable, in fact. Unlike in many other applications of optimal (stochastic) control and calculus of variations, the pertinent objective functions in economics and finance - the utility functions - are too "badly-behaved" to allow for the well-developed and classical $\mathbb{L}^p$ theory (with $p \in [1, \infty]$) to be applied. Moreover, the budget-type constraints present in incomplete markets are typically posed with respect to a class of probability measures (which are often the martingale measures for a particular financial model) and not with respect to a single measure, as often required by the $\mathbb{L}^p$ theory. The immediate consequence of this fact is that the only

natural functional-analytic framework must be, in a sense, measure-free. In the hierarchy of the $\mathbb{L}^p$ spaces, this leaves the two extremes: $\mathbb{L}^0$ and $\mathbb{L}^\infty$. The smaller of the two - $\mathbb{L}^\infty$ - is clearly too small to contain even the most important special cases (the solution to the Merton's problem, for example), so we are forced to work with $\mathbb{L}^0$. The price that needs to be paid is the renouncement of a large number of classical functional-analytic tools which were developed with locally-convex spaces (and, in particular, Banach spaces) in mind. $\mathbb{L}^0$ fails the local-convexity property in a dramatic fashion: if $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic, the topological dual of $\mathbb{L}^0$ is trivial, i.e., equals $\{0\}$. Therefore, a new set of functional-analytic tools which do not rely on local convexity are needed to treat even the most basic problems posed by finance and economics: pricing, optimal investment, and, ultimately, existence of equilibria.

In addition to existence results for the usual optimization problems in quantitative finance (see Merton (1969, 1971), Pliska (1986), Cox and Huang (1989), Karatzas et al. (1990), He and Pearson (1991), Kramkov and Schachermayer (1999), Cvitanić et al. (2001), Karatzas and Žitković (2003)), some recent results dealing with stability properties of those problems testify even stronger to the indispensability of $\mathbb{L}^0_+$ (and the related convergence in probability) in this context. As Larsen and Žitković (2007) and Kardaras and Žitković (2007) have shown, the output (i.e., the demand function) of such optimization procedures is a continuous transformation of the primitives (a parametrization of a set of models) when both are viewed as subsets of $\mathbb{L}^0_+$. Moreover, there are simple examples showing that such continuity fails if the space $\mathbb{L}^0_+$ is upgraded to an $\mathbb{L}^p$-space, for $p \geq 1$. In addition to their intrinsic interest, such stability is an important ingredient in the study of equilibria in incomplete financial (stochastic) models. Indeed, the continuity in probability of the excess-demand function enters as condition *1.* in Theorem 4.11. Let us also mention that the most relevant classical approaches to the general-equilibrium theory (see Bewley (1972) or Mas-Colell and Zame (1991)) depend on the utility functions exhibiting a certain degree of continuity in the Mackey topology. This condition is not satisfied in general in the Alt-von Neumann-Morgenstern setting for utility functions satisfying the Inada conditions (the HARA and CRRA families, e.g.). They do give rise to $\mathbb{L}^0_+$-continuous demand functions under mild regularity conditions.

Another contribution of the present paper is the realization that, apart from its applicability to $\mathbb{L}^0_+$, convex compactness is a sufficient condition for several well-known and widely-applicable optimization results to hold under very general conditions. We start with a simple observation that lower semicontinuous, quasiconvex and appropriately coercive functionals attain their infima on convexly compact sets. In the same spirit, we establish a version of the Minimax Theorem where compactness is replaced by convex compactness.

The central application of convex compactness is a generalization of the classical "fixed-point" theorem of Knaster, Kuratowski and Mazurkiewicz (see Knaster et al. (1929)) to convexly compact sets. In addition to being interesting in its own right, it is used to give a variant of the Walrasian excess-demand theorem of general-equilibrium theory based on convex compactness. This result can serve as a theoretical basis for various equilibrium existence theorems where regularity of the primitives implies only weak regularity (continuity in probability) of the excess-demand functions. As already mentioned above, that is, for instance, the case when the agents'

preferences are induced by Alt-von Neumann-Morgenstern expected utilities which satisfy Inada conditions, but are not necessarily Mackey-continuous.

The paper is structured as follows: after this introduction, section 2. defines the notion of convex compactness and provides necessary tools for its characterization. Section 3. deals with the main source of examples of convex compactness - the space $\mathbb{L}^0_+$ of all non-negative random variables. The applications are dealt with in section 4.

2 Convex compactness

2.1 The notion of convex compactness

Let A be a non-empty set. The set $\mathrm{Fin}(A)$ consisting of all non-empty finite subsets of A carries a natural structure of a partially ordered set when ordered by inclusion. Moreover, it is a directed set, since $D_1, D_2 \subseteq D_1 \cup D_2$ for any $D_1, D_2 \in \mathrm{Fin}(A)$.

Definition 2.1. A convex subset C of a topological vector space X is said to be *convexly compact* if for any non-empty set A and any family $\{F_\alpha\}_{\alpha\in A}$ of closed and convex subsets of C, the condition

$$\forall\, D \in \mathrm{Fin}(A),\ \bigcap_{\alpha\in D} F_\alpha \neq \emptyset \tag{2.1}$$

implies

$$\bigcap_{\alpha\in A} F_\alpha \neq \emptyset. \tag{2.2}$$

Without the additional restriction that the sets $\{F_\alpha\}_{\alpha\in A}$ be convex, Definition 2.1 - postulating the finite-intersection property for families of closed and convex sets - would be equivalent to the classical definition of compactness. It is, therefore, immediately clear that any convex and compact subset of a topological vector space is convexly compact.

Example 2.2 (Convex compactness without compactness). Let L be a locally-convex topological vector space, and let L^* be the topological dual of L, endowed with some compatible topology τ, possibly different from the weak-* topology $\sigma(L^*, L)$. For a neighborhood N of 0 in L, define the set C in the topological dual L^* of L by

$$C = \{x^* \in L \,:\, \langle x, x^*\rangle \leq 1,\ \forall\, x \in N\}.$$

In other words, $C = N^\circ$ is the polar of N. By the Banach-Alaoglu Theorem, C is compact with respect to the weak-* topology $\sigma(L^*, L)$, but it may not be compact with respect to τ. On the other hand, let $\{F_\alpha\}_{\alpha\in A}$ be a non-empty family of convex and τ-closed subsets of C with the finite-intersection property (2.1). It is a classical consequence of the Hahn-Banach Theorem that the collection of closed and convex sets is the same for all topologies consistent with a given dual pair. Therefore, the sets $\{F_\alpha\}_{\alpha\in A}$ are $\sigma(L^*, L)$-closed, and the relation (2.2) holds by the aforementioned $\sigma(L^*, L)$-compactness of C.

2.2 A characterization in terms of generalized sequences

The classical theorem of Komlós (1967) states that any norm-bounded sequence in $\mathbb{L}^1$ admits a subsequence whose Cesàro sums converge a.s. The following characterization draws a parallel between Komlós' theorem and the notion of convex compactness. It should be kept in mind that we forgo equal (Cesáro) weights guaranteed by Komlós' theorem, and settle for generic convex combinations. We remind the reader that for a subset C of a vector space X, $\operatorname{conv} C$ denotes the smallest convex subset of X containing C.

Definition 2.3. Let $\{x_\alpha\}_{\alpha\in A}$ be a net in a vector space X. A net $\{y_\beta\}_{\beta\in B}$ is said to be a *subnet of convex combinations of* $\{x_\alpha\}_{\alpha\in A}$ if there exists a mapping $D : B \to \operatorname{Fin}(A)$ such that

1. $y_\beta \in \operatorname{conv}\{x_\alpha \,:\, \alpha \in D(\beta)\}$ for each $\beta \in B$, and
2. for each $\alpha \in A$ there exists $\beta \in B$ such that $\alpha' \succeq \alpha$ for each $\alpha' \in \bigcup_{\beta'\succeq\beta} D(\beta')$.

Proposition 2.4. *A closed and convex subset C of a topological vector space X is convexly compact if and only if for any net $\{x_\alpha\}_{\alpha\in A}$ in C there exists a subnet $\{y_\beta\}_{\beta\in B}$ of convex combinations of $\{x_\alpha\}_{\alpha\in A}$ such that $y_\beta \to y$ for some $y \in C$.*

Proof. $\Rightarrow$ Suppose, first, that C is convexly compact, and let $\{x_\alpha\}_{\alpha\in A}$ be a net in C. For $\alpha \in A$ define the closed and convex set $F_\alpha \subseteq C$ by

$$F_\alpha = \overline{\operatorname{conv}}\{x_{\alpha'} \,:\, \alpha' \succeq \alpha\},$$

where, for any $G \subseteq X$, $\overline{\operatorname{conv}}\, G$ denotes the closure of $\operatorname{conv} G$. By convex compactness of C, there exists $y \in \cap_{\alpha\in A} F_\alpha$. Define $B = \mathcal{U} \times A$, where $\mathcal{U}$ is the collection of all neighborhoods of y in X. The binary relation $\preceq_B$ on B, defined by

$$(U_1, \alpha_1) \preceq_B (U_2, \alpha_2) \text{ if and only if } U_2 \subseteq U_1 \text{ and } \alpha_1 \preceq \alpha_2,$$

is a partial order under which B becomes a directed set.

By the construction of the family $\{F_\alpha\}_{\alpha\in A}$, for each $\beta = (U, \alpha) \in B$ there exists a finite set $D(\beta)$ such that $\alpha' \succeq \alpha$ for each $\alpha' \in D(\beta)$ and an element $y_\beta \in \operatorname{conv}\{x_{\alpha'} \,:\, \alpha' \in D(\beta)\} \cap U$. It is evident now that $\{y_\beta\}_{\beta\in B}$ is a subnet of convex combinations of $\{x_\alpha\}_{\alpha\in A}$ which converges towards y.

$\Leftarrow$ Let $\{F_\alpha\}_{\alpha\in A}$ be a family of closed and convex subsets of C satisfying (2.1). For $\delta \in \operatorname{Fin}(A)$, we set

$$G_\delta = \bigcap_{\alpha\in\delta} F_\alpha \neq \emptyset,$$

so that $\{G_\delta\}_{\delta\in\operatorname{Fin}(A)}$ becomes an non-increasing net in $(2^C, \subseteq)$, in the sense that $G_{\delta_1} \subseteq G_{\delta_2}$ when $\delta_1 \supseteq \delta_2$. For each $\delta \in \operatorname{Fin}(A)$, we pick $x_\delta \in G_\delta$. By the assumption, there exists a subnet $\{y_\beta\}_{\beta\in B}$ of convex combinations of $\{x_\delta\}_{\delta\in\operatorname{Fin}(A)}$ converging to some $y \in C$. More precisely, in the present setting, conditions (1) and (2) from Definition 2.3 yield the existence of a directed set B and a function $D : B \to \operatorname{Fin}(\operatorname{Fin}(A))$ such that:

(1) $y_\beta \in \operatorname{conv}\{x_\delta : \delta \in D(\beta)\} \subseteq \operatorname{conv} \cup_{\delta \in D(\beta)} G_\delta$; in particular, $y_\beta \in G_\delta$ for each δ such that $\delta \subseteq \cap_{\delta' \in D(\beta)} \delta'$.

(2) For each $\delta \in \operatorname{Fin}(A)$ there exists $\beta \in B$ such that $\delta \subseteq \delta'$ for all $\delta' \in \cup_{\beta' \succeq \beta} D(\beta')$. In particular, if we set $\delta = \{\alpha\}$ for an arbitrary $\alpha \in A$, we can assert the existence of $\beta_\alpha \in B$ such that $\alpha \in \delta'$ for any $\delta' \in \cup_{\beta' \succeq \beta_\alpha} D(\beta')$.

Combining (1) and (2) above, we get that for each $\alpha \in A$ there exists $\beta_\alpha \in B$ such that $y_\beta \in G_{\{\alpha\}} = F_\alpha$ for all $\beta \succeq \beta_\alpha$. Since $y_\beta \to y$ and F_α is a closed set, we necessarily have $y \in F_\alpha$ and, thus, $y \in \cap_{\alpha \in A} F_\alpha$. Therefore, $\cap_{\alpha \in A} F_\alpha \neq \emptyset$. □

3 A space of random variables

3.1 Convex compactness in $\mathbb{L}^0_+$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The positive orthant $\{f \in \mathbb{L}^0 : f \geq 0, \text{ a.s.}\}$ will be denoted by $\mathbb{L}^0_+$. A set $A \subseteq \mathbb{L}^0$ is said to be *bounded in* $\mathbb{L}^0$ (or *bounded in probability*) if

$$\lim_{M \to \infty} \sup_{f \in A} \mathbb{P}[|f| \geq M] = 0.$$

Unless specified otherwise, any mention of convergence on $\mathbb{L}^0$ will be under the topology of convergence in probability, induced by the translation-invariant metric $d(f,g) = \mathbb{E}[1 \wedge |f-g|]$, making $\mathbb{L}^0$ a Fréchet space (a topological vector space admitting a complete compatible metric). It is well-known, however, that $\mathbb{L}^0$ is generally *not* a locally-convex topological vector space. In fact, when $\mathbb{P}$ is non-atomic, $\mathbb{L}^0$ admits no non-trivial continuous linear functionals (see Kalton et al. (1984), Theorem 2.2, p. 18).

In addition to not being locally-convex, the space $\mathbb{L}^0$ has poor compactness properties. Indeed, let C be a closed uniformly-integrable ($\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$-compact) set in $\mathbb{L}^1 \subseteq \mathbb{L}^0$. By a generalization of the dominated convergence theorem, the topology of convergence in probability and the strong $\mathbb{L}^1$-topology on C coincide; a statement of this form is sometimes know as Osgood's theorem (see, e.g., Theorem 16.14, p. 127, in Billingsley (1995)). Therefore, the question of $\mathbb{L}^0$-compactness reduces to the question of strong $\mathbb{L}^1$-compactness for such sets. It is a well-known fact that C is $\mathbb{L}^1$-compact (equivalently, $\mathbb{L}^0$-compact) if and only if it satisfies a non-trivial condition of Girardi (1991) called the *Bocce criterion.* The following result shows that, however, a much larger class of sets shares the convex-compactness property. In fact, there is a direct analogy between the present situation and the well-known characterization of compactness in Euclidean spaces.

Theorem 3.1. *A closed and convex subset C of $\mathbb{L}^0_+$ is convexly compact if and only if it is bounded in probability.*

Proof. $\Leftarrow$ Let C be a convex, closed and bounded-in-probability subset of $\mathbb{L}^0_+$, and let $\{F_\alpha\}_{\alpha \in A}$ be a family of closed and convex subsets of C satisfying (2.1). For $D \in \operatorname{Fin}(A)$ we define

$$G_D = C, \text{ when } D = \emptyset, \text{ and } G_D = \cap_{\alpha \in D} F_\alpha, \text{ otherwise,}$$

and fix an arbitrary $f_D \in G_D$. With $\varphi(x) = 1 - \exp(-x)$, we set

$$u_D = \sup\{\mathbb{E}[\varphi(g)] \,:\, g \in \operatorname{conv}\{f_{D'} \,:\, D' \supseteq D\}\},$$

so that $0 \leq u_D \leq 1$ and $u_{D_1} \geq u_{D_2}$, for $D_1 \subseteq D_2$. Seen as a net on the directed set $(\operatorname{Fin}(A), \subseteq)$, $\{u_D\}_{D \in \operatorname{Fin}(A)}$ is monotone and bounded, and therefore convergent, i.e., $u_D \to u_\infty$, for some $u_\infty \in [0,1]$. Moreover, for each $D \in \operatorname{Fin}(A)$ we can choose $g_D \in \operatorname{conv}\{g_{D'} \,:\, D \subseteq D'\}$ so that

$$u_D \geq \gamma_D \triangleq \mathbb{E}[\varphi(g_D)] \geq u_D - \frac{1}{\#D},$$

where $\#D$ denotes the number of elements in D. Clearly, $\gamma_D \to u_\infty$.

The reader is invited to check that simple analytic properties of the function φ are enough to prove the following statement: for each $M > 0$ there exists $\varepsilon = \varepsilon(M) > 0$, such that

$$\text{if } \; x_1, x_2 \geq 0,\; |x_1 - x_2| \geq \tfrac{1}{M} \;\text{ and }\; 0 \leq \min(x_1, x_2) \leq M,$$
$$\text{then } \varphi(\tfrac{1}{2}(x_1 + x_2)) \geq \tfrac{1}{2}(\varphi(x_1) + \varphi(x_2)) + \varepsilon.$$

It follows that for any $D_1, D_2 \in \operatorname{Fin}(A)$ we have

$$\varepsilon\mathbb{P}\Big[\,|g_{D_1} - g_{D_2}| \geq \tfrac{1}{M},\; \min(g_{D_1}, g_{D_2}) \leq M\Big] \leq$$
$$\leq \mathbb{E}\left[\varphi\Big(\tfrac{1}{2}(g_{D_1} + g_{D_2})\Big)\right] - \tfrac{1}{2}\left(\mathbb{E}[\varphi(g_{D_1})] + \mathbb{E}[\varphi(g_{D_2})]\right).$$

The random variable $\frac{1}{2}(g_{D_1} + g_{D_2})$ belongs to $\operatorname{conv}\{f_{D'} \,:\, D' \supseteq D_1 \cap D_2\}$, so

$$\mathbb{E}[\varphi(\tfrac{1}{2}(g_{D_1} + g_{D_2}))] \leq u_{D_1 \cap D_2}.$$

Consequently,

$$0 \leq \varepsilon\mathbb{P}[|g_{D_1} - g_{D_2}| \geq 1/M,\; \min(g_{D_1}, g_{D_2}) \leq M] \leq \eta_{D_1, D_2},$$

where $\eta_{D_1,D_2} = u_{D_1 \cap D_2} - \frac{1}{2}(u_{D_1} + u_{D_2}) + \frac{1}{2}(\frac{1}{\#D_1} + \frac{1}{\#D_2})$. Thanks to the boundedness in probability of the set C, for $\kappa > 0$, we can find $M = M(\kappa) > 0$ such that $M > 1/\kappa$ and $\mathbb{P}[f \geq M] < \kappa/2$ for any $f \in C$. Furthermore, let $D(\kappa) \in \operatorname{Fin}(A)$ be such that $u_\infty + \varepsilon(M)\kappa/4 \geq u_D \geq u_\infty$ for all $D \supseteq D(\kappa)$, and $\#D(\kappa) > 4/(\varepsilon(M)\kappa)$. Then, for $D_1, D_2 \supseteq D(\kappa)$ we have

$$\begin{aligned}\mathbb{P}[|g_{D_1} - g_{D_2}| \geq \kappa] &\leq \mathbb{P}[|g_{D_1} - g_{D_2}| \geq \tfrac{1}{M}, \min(g_{D_1}, g_{D_2}) \leq M] + \mathbb{P}[\min(g_{D_1}, g_{D_2}) \geq M] \\ &\leq \tfrac{1}{\varepsilon(M)}\Big(u_{D_1 \cap D_2} - \tfrac{1}{2}\big(u_{D_1} + u_{D_2}\big) + \tfrac{1}{2}\big(\tfrac{1}{\#D_1} + \tfrac{1}{\#D_2}\big)\Big) \leq \kappa.\end{aligned}$$

In other words, $\{g_D\}_{D \in \operatorname{Fin}(A)}$ is a Cauchy net in $\mathbb{L}^0_+$ which, by completeness, admits a limit $g_\infty \in \mathbb{L}^0_+$. By construction and convexity of the sets F_α, $\alpha \in A$, we have $g_D \in F_\alpha$ whenever $D \supseteq \{\alpha\}$. By closedness of F_α, we conclude that $g_\infty \in F_\alpha$, and so, $g_\infty \in \cap_{\alpha \in A} F_\alpha$.

$\Rightarrow$ It remains to show that convexly compact sets in $\mathbb{L}^0_+$ are necessarily bounded in probability. Suppose, to the contrary, that $C \subseteq \mathbb{L}^0_+$ is convexly compact, but not bounded in probability. Then, there exists a constant $\varepsilon \in (0,1)$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ in C such that

$$\mathbb{P}[f_n \geq n] > \varepsilon, \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

By Proposition 2.4, there exists a subnet $\{g_\beta\}_{\beta\in B}$ of convex combinations of $\{f_n\}_{n\in\mathbb{N}}$ which converges to some $g \in C$. In particular, for each $n \in \mathbb{N}$ there exists $\beta_n \in B$ such that g_β can be written as a finite convex combination of the elements of the set $\{f_m : m \geq n\}$, for any $\beta \succeq \beta_n$. Using (3.1) and Lemma 9.8.6., p. 205 in Delbaen and Schachermayer (2006), we get the following estimate

$$\mathbb{P}[g_\beta \geq \tfrac{n\varepsilon}{2}] \geq \tfrac{\varepsilon}{2}, \text{ for all } \beta \succeq \beta_n. \tag{3.2}$$

Therefore,

$$\mathbb{P}[g \geq \tfrac{n\varepsilon}{4}] \geq \mathbb{P}[g_\beta \geq \tfrac{n\varepsilon}{2}] - \mathbb{P}[|g - g_\beta| > \tfrac{n\varepsilon}{4}] \geq \tfrac{\varepsilon}{2} - \mathbb{P}[|g - g_\beta| > \tfrac{\varepsilon}{4}] > \tfrac{\varepsilon}{4},$$

for all "large enough" $\beta \in B$. Hence, $\mathbb{P}[g = +\infty] > 0$ - a contradiction with the assumption $g \in C$. □

4 Applications

Substitution of the strong notion of compactness for a weaker notion of convex compactness opens a possibility for extensions of several classical theorems to a more general setting. In the sequel, let X denote a generic topological vector space.

4.1 Attainment of infima for convexly coercive functions

We set off with a simple claim that convex and appropriately regular functionals attain their infima on convex-compact sets. This fact (in a slightly different form) has been observed and used, e.g., in Kramkov and Schachermayer (1999).

For a function $G : X \to (-\infty, \infty]$ and $\lambda \in (-\infty, \infty]$, we define the λ-*lower-contour set* $L_G(\lambda)$ as $L_G(\lambda) = \{x \in X : G(x) \leq \lambda\}$. The *effective domain* $\mathrm{Dom}(G)$ of G is defined as $\mathrm{Dom}(G) = \cup_{\lambda<\infty} L_G(\lambda)$.

Definition 4.1. A function $G : X \to (-\infty, \infty]$ is said to be *convexly coercive* if $L_G(\lambda)$ is convex and closed for all $\lambda \in (-\infty, \infty]$, and there exists $\lambda_0 \in (-\infty, \infty]$ such that $L_G(\lambda_0)$ is non-empty and convexly compact.

Remark 4.2. By Theorem 3.1, in the special case when $X \subseteq \mathbb{L}^0$, and $\mathrm{Dom}(G) \subseteq \mathbb{L}^0_+$, convex coercivity is implied by the conjunction of the following three conditions:

1. *weak coercivity:* there exists $\lambda_0 \in (-\infty, \infty]$ such that the $L_G(\lambda_0)$ is bounded-in-probability,
2. *lower semi-continuity:* $L_G(\lambda)$ is closed for each $\lambda \in (-\infty, \infty)$, and
3. *quasi-convexity:* $L_G(\lambda)$ is convex for each $\lambda_0 \in (-\infty, \infty)$ (a condition automatically satisfied when G is convex).

Lemma 4.3. *Each bounded-from-below convexly coercive function $G : X \to (-\infty, \infty]$ attains its infimum on X.*

Proof. Let $r_0 = \inf\{G(x) : x \in X\}$. Let $a_0 = G(x_0)$, for some $x_0 \in L_G(\lambda_0)$, with λ_0 as in Definition 4.1 above. If $a_0 = r_0$, we are done. Suppose, therefore, that $a_0 > r_0$. For $a \in (r_0, a_0]$ we define

$$F_a = \{x \in X : G(x) \leq a_0\} \neq \emptyset.$$

The family $\{F_a : a \in (r_0, a_0]\}$ of convex and closed subsets of the convex-compact set $L_G(\lambda_0)$ is nested; in particular, it has the finite intersection property (2.1). Therefore, there exists $x_0^* \in \cap_{a\in(r_0,a_0]} F_a$. It follows that $G(x_0^*) = r_0$, and so the minimum of G is attained at x_0^*. □

Remark 4.4. Let $\Phi : [0, \infty) \to [0, \infty)$ be a convex and lower semi-continuous function. Define the mapping $G : \mathbb{L}^0_+ \to [0, \infty]$ by

$$G(f) = \mathbb{E}[\Phi(f)], \ f \in \mathbb{L}^0_+.$$

G is clearly convex and Fatou's lemma implies that it is lower semi-continuous. In order to guarantee its convex coercivity, one can either restrict the function G onto a convex, closed and bounded-in-probability subset B of $\mathbb{L}^0$ (by setting $G(f) = +\infty$ for $f \in B^c$), or impose growth conditions on the function Φ. A simple example of such a condition is the following (which is, in fact, equivalent to $\lim_{x\to\infty} \Phi(x) = +\infty$):

$$\liminf_{x\to\infty} \frac{\Phi(x)}{x} > 0. \tag{4.1}$$

Indeed, if (4.1) holds, then there exist constants $D \in \mathbb{R}$ and $\delta > 0$ such that $\Phi(x) \geq D + \delta x$. Therefore, for $f_0 \equiv 1$, with $c = G(f_0) = \Phi(1)$ we have

$$\{f \in \mathbb{L}^0_+ : G(f) \leq c\} \subseteq \{f \in \mathbb{L}^0_+ : \mathbb{E}[f] \leq (c - D)/\delta\}.$$

This set on the right-hand side above is bounded in $\mathbb{L}^1$ and, therefore, in probability.

4.2 A version of the theorem of Knaster, Kuratowski and Mazurkiewicz

The celebrated theorem of Knaster, Kuratowski and Mazurkiewicz originally stated for finite-dimensional simplices, is commonly considered as a mathematical basis for the general equilibrium theory of mathematical economics.

Theorem 4.5 (Knaster, Kuratowski and Mazurkiewicz (1929))**.** *Let S be the unit simplex in $\mathbb{R}^d$, $d \geq 1$, with vertices $x_1, x_2, \ldots, x_d$. Let $\{F_i : i = 1, \ldots, d\}$, be a collection of closed subsets of S such that*

$$\operatorname{conv}\{x_{i_1}, \ldots, x_{i_k}\} \subseteq \cup_{j=1}^k F_{i_j} \text{ for any subset } \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}.$$

Then,

$$\bigcap_{i=1}^d F_i \neq \emptyset.$$

Before stating a useful, yet simple, corollary to Theorem 4.5, we introduce the KKM-property:

Definition 4.6. Let X be a vector space and let B be its non-empty subset. A family $\{F(x)\}_{x \in B}$ of subsets of X is said to have the **Knaster-Kuratowski-Mazurkiewicz (KKM) property** if

1. $F(x)$ is closed for each $x \in B$, and

2. $\operatorname{conv}\{x_1, \dots, x_n\} \subseteq \cup_{i=1}^n F(x_i)$, for any finite set $\{x_1, \dots, x_n\} \in B$.

Corollary 4.7. *Let X be a vector space, and let B be a non-empty subset of X. Suppose that a family $\{F(x)\}_{x \in B}$ of subsets of X has the KKM property. Then,*

$$\bigcap_{i=1}^{n} F(x_i) \neq \emptyset,$$

for any finite set $\{x_1, \dots, x_n\} \subseteq B$.

The literature on fixed points abounds with extensions of Theorem 4.5 to various locally-convex settings (see, e.g., Chapter I, §4 in Granas and Dugundji (2003)). To the best of our knowledge, this extension has not been made to the class of Fréchet spaces (and $\mathbb{L}^0$ in particular). The following result is a direct consequence of the combination of Theorem 4.5 and Theorem 3.1. The reader can observe that the usual assumption of compactness has been replaced by convexity and convex compactness.

Theorem 4.8. *Let X be a topological vector space, and let B be a non-empty subset of X. Suppose that a family $\{F(x)\}_{x \in B}$ of convex subsets of X, indexed by B, has the KKM property. If there exists $x_0 \in B$ such that $F(x_0)$ is convexly compact, then*

$$\bigcap_{x \in B} F(x) \neq \emptyset.$$

Proof. Define $\tilde{F}(x) = F(x) \cap F(x_0)$, for $x \in B$. By Corollary 4.7, any finite subfamily of $\{\tilde{F}(x)\}_{x \in B}$ has a non-empty intersection. Since, all $\tilde{F}(x)$, $x \in B$ are convex and closed subsets of the convexly compact set $F(x_0)$, we have

$$\cap_{x \in B} F(x) = \cap_{x \in B} \tilde{F}(x) \neq \emptyset.$$

□

4.3 A minimax theorem for $\mathbb{L}^0_+$

Since convexity already appears naturally in the classical Minimax theorem, one can replace the usual compactness by convex compactness at little cost.

Theorem 4.9. *Let C, D be two convexly compact (closed, convex and bounded-in-probability) subsets of $\mathbb{L}^0_+$, and let $\Phi : C \times D \to \mathbb{R}$ be a function with the following properties:*

1. *$f \mapsto \Phi(f, g)$ is concave and upper semi-continuous for all $g \in D$,*

2. $g \mapsto \Phi(f,g)$ *is convex and lower semi-continuous for all* $f \in C$.

Then, there exists a pair $(f_0, g_0) \in C \times D$ *such that* (f_0, g_0) *is a saddle-point of* Φ*, i.e.,*

$$\Phi(f, g_0) \leq \Phi(f_0, g_0) \leq \Phi(f_0, g), \text{ for all } (f,g) \in C \times D.$$

Moreover,

$$\sup_{f \in C} \inf_{g \in D} \Phi(f,g) = \inf_{g \in D} \sup_{f \in C} \Phi(f,g). \tag{4.2}$$

Proof. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ be a direct sum of $(\Omega, \mathcal{F}, \mathbb{P})$ and a copy of itself. More precisely, we set $\hat{\Omega} = \{1,2\} \times \Omega$; $\hat{\mathcal{F}}$ is the σ-algebra on $\hat{\Omega}$ generated by the sets of the form $\{i\} \times A$, $i = 1, 2$, $A \in \mathcal{F}$, and $\hat{\mathbb{P}}$ the unique probability measure on $\hat{\mathcal{F}}$ satisfying $\hat{\mathbb{P}}[\{i\} \times A] = \frac{1}{2}\mathbb{P}[A]$, for $i = 1, 2$ and $A \in \mathcal{F}$. Then, a pair (f,g) in $\mathbb{L}^0_+(\Omega, \mathcal{F}, \mathbb{P}) \times \mathbb{L}^0_+(\Omega, \mathcal{F}, \mathbb{P})$ can be identified with the element $f \oplus g$ of $\hat{\mathbb{L}}^0_+ = \mathbb{L}^0_+(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ in the following way: $(f \oplus g)(i, \omega) = f(\omega)$ if $i = 1$, and $(f \oplus g)(i, \omega) = g(\omega)$ if $i = 2$.

For $f \oplus g \in \hat{\mathbb{L}}^0_+$, define $C \oplus D = \{f \oplus g \in \hat{\mathbb{L}}^0_+ \,:\, (f,g) \in C \times D\}$, together with the family of its subsets

$$G_{f \oplus g} = \{f' \oplus g' \in C \oplus D \,:\, \Phi(f, g') - \Phi(f', g) \leq 0\}, \; f \oplus g \in C \oplus D.$$

By properties *1.* and *2.* of the function Φ in the statement of the theorem, the set $G_{f \oplus g}$ is a convex subset of $C \oplus D$. Moreover, $f_n \oplus g_n \to f \oplus g$ in $\hat{\mathbb{L}}^0$ if and only if $f_n \to f$ and $g_n \to g$ in $\mathbb{L}^0$. Therefore, we can use upper semi-continuity of the maps $\Phi(\cdot, g)$ and $-\Phi(f, \cdot)$, valid for any $f \in C$ and $g \in D$, to conclude that $G_{f \oplus g}$ is closed in $\hat{\mathbb{P}}$ for each $f \oplus g \in C \oplus D$. Finally, since $\hat{\mathbb{P}}[f \oplus g \geq M] = \frac{1}{2}(\mathbb{P}[f \geq M] + \mathbb{P}[g \geq M])$, it is clear that $C \oplus D$ is bounded in probability. Therefore, $G_{f \oplus g}$ is a family of closed and convex subsets of a convex-compact set $C \oplus D$. Next, we state and prove an auxiliary claim.

Claim 4.10. The family $\{G_{f \oplus g} \,:\, f \oplus g \in C \oplus D\}$ has the KKM property.

To prove Claim 4.10, we assume, to the contrary, that there exist $\tilde{f} \oplus \tilde{g} \in C \oplus D$, a finite family $f_1 \oplus g_1, \dots, f_m \oplus g_m$ in $C \oplus D$ and a set of non-negative weights $\{\alpha_k\}_{k \in 1, \dots, m}$ with $\sum_{k=1}^m \alpha_k = 1$, such that $\tilde{f} \oplus \tilde{g} = \sum_{k=1}^m \alpha_k \, f_k \oplus g_k$, but $\tilde{f} \oplus \tilde{g} \notin G_{f_k \oplus g_k}$ for $k = 1, \dots, m$, i.e., $\Phi(f_k, \tilde{g}) > \Phi(\tilde{f}, g_k)$, for $k = 1, \dots, m$. Then, we have the following string of inequalities:

$$\Phi(\tilde{f}, \tilde{g}) \geq \sum_{k=1}^m \alpha_k \Phi(f_k, \tilde{g}) > \sum_{k=1}^n \alpha_k \Phi(\tilde{f}, g_k) \geq \Phi(\tilde{f}, \tilde{g}).$$

Evidently, this is a contradiction. Therefore, Claim 4.10 is established.

We continue the proof invoking of Theorem 4.8; its assumptions are satisfied, thanks to Claim 4.10 and the discussion preceding it. By Theorem 4.8, there exists $f_0 \oplus g_0 \in C \oplus D$ such that $(f_0, g_0) \in G_{f \oplus g}$ for all $f \in C$ and all $g \in D$, i.e., $\Phi(f, g_0) \leq \Phi(f_0, g)$ for all $f \in C$ and $g \in D$. Substituting $f = f_0$ or $g = g_0$, we obtain

$$\Phi(f, g_0) \leq \Phi(f_0, g_0) \leq \Phi(f_0, g), \text{ for all } f \in C, \, g \in D.$$

The last statement of the theorem - equation (4.2) - follows readily. $\square$

4.4 An excess-demand theorem under convex compactness

Our last application is an extension of the Walrasian excess-demand theorem. Similarly to the situations described above, versions of excess-demand theorem have been proved in various settings (see, e.g., Arrow and Debreu (1954), McKenzie (1959) and Exercise C.7, p. 179 in Granas and Dugundji (2003)), but, to the best of our knowledge, always under the assumption of local convexity and compactness.

Typically, the excess-demand theorem is applied to a function F of the type $F(x,y) = \langle \Delta(x), y\rangle$, where x is thought of as a price-system, $\Delta(x)$ is the excess aggregate demand (or a zero-preserving transformation thereof) for the bundle of all commodities, and y is a test function. The conclusion $F(x_0, y) \leq 0$, for every y, is then used to establish the equality $\Delta(x_0) = 0$ - a stability (equilibrium) condition for the market under consideration.

Theorem 4.11. *Let C be a convexly compact subset of a topological vector space X, and let $D \subseteq C$ be convex and closed. Let the mapping $F : C \times D \to \mathbb{R}$ satisfy the following properties*

1. *for each $y \in D$, the set $\{x \in C : F(x,y) \leq 0\}$ is closed and convex,*
2. *for each $x \in C$, the function $y \mapsto F(x,y)$ is concave, and*
3. *for each $y \in D$, $F(y,y) \leq 0$.*

Then there exists $x_0 \in C$ such that $F(x_0, y) \leq 0$, for all $y \in D$.

Proof. Define the family $\{F_y\}_{y\in D}$ of closed and convex subsets of C by $F_y = \{x \in C : F(x,y) \leq 0\}$. In order to show that it has the KKM property, we assume, to the contrary, that there exist $y_1, \dots, y_m$ in D and a set of non-negative weights $\alpha_1, \dots, \alpha_m$ with $\sum_{k=1}^m \alpha_k = 1$ such that $\tilde{y} := \sum_{k=1}^m \alpha_k y_k \not\in \cup_{k=1}^m F_{y_k}$, i.e., $F(\tilde{y}, y_k) > 0$, for $k = 1, \dots, m$. Then, by property *2.* of F, we have $0 \geq F(\tilde{y}, \tilde{y}) \geq \sum_{k=1}^m \alpha_k F(\tilde{y}, y_k) > 0$ – a contradiction. Therefore, by Theorem 4.8, there exists $x_0 \in C$ such that $x_0 \in F_y$ for all $y \in D$, i.e., $F(x_0, y) \leq 0$, for all $y \in D$. □

It is outside the scope of the present paper to construct in detail concrete equilibrium situations where the excess-demand function above is used to guarantee the existence of a Walrasian equilibrium - this will be a topic of our future work.

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